NCERT Solutions Class 12 Maths Chapter 1 Exercise 1.4

Question 1:

Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.

i. On $\mathbf{Z}^+$, define $*$ by $a*b=a-b$
ii. On $\mathbf{Z}^+$,define $*$ by $a*b=ab$
iii. On $\mathbf{R}$, define $*$by $a*b=ab^2$
iv. On $\mathbf{Z}^+$, define $*$ by $a*b=|a-b|$
v. On $\mathbf{Z}^+$, define $*$ by $a*b=a$

Solution:

i. On $\mathbf{Z}^+$, define $*$ by $a*b=a-b$

It is not a binary operation as the image of $(1,2)$ under $*$ is

$1*2=1-2$

$\Rightarrow -1 \notin \mathbf{Z}^+$.

Therefore, $*$ is not a binary operation.

ii. On $\mathbf{Z}^+$,define $*$ by $a*b=ab$

It is seen that for each $a,b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that $*$ carries each pair (a,b) to a unique element $a*b=ab$ in $\mathbf{Z}^+$.

Therefore, $*$ is a binary operation.

iii. On $\mathbf{R}$, define $*$ $a*b=ab^2$

It is seen that for each $a,b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$. This means that $*$ carries each pair (a,b) to a unique element $a*b=ab^2$ in **R**.

Therefore, $*$is a binary operation.

iv. On $\mathbf{Z}^+$, define $*$ by $a*b=|a-b|$

It is seen that for each $a,b \in \mathbf{Z}^+$,there is a unique element $|a-b|$ in $\mathbf{Z}^+$. This means that $*$ carries each pair (a,b) to a unique element $a*b=|a-b|$ in $\mathbf{Z}^+$. Therefore, $*$is a binary operation.

v. On $\mathbf{Z}^+$, define $*$ by $a*b=a$

$*$carries each pair (a, b) to a unique element in $a*b=a$ in $\mathbf{Z}^+$.

Therefore, $*$ is a binary operation.

Question 2:

For each binary operation $*$defined below, determine whether $*$ is commutative or associative.

i. On $\mathbf{Z}^+$, define $a*b=a-b$

ii. On $\mathbf{Q}$, define $a*b=ab+1$

iii. On $\mathbf{Q}$, define $a*b=\frac{ab}{2}$

iv. On $\mathbf{Z}^{+}$, define $a*b=2^{ab}$

v. On $\mathbf{Z}^{+}$, define $a*b=a^{b}$

vi. On $\mathbf{R}-\{-1\}$, define $a*b=\frac{a}{b+1}$

Solution:

i. On $\mathbf{Z}^{+}$, define $a*b=a-b$

It can be observed that $1*2=1-2=-1$ and $2*1=2-1=1$.

$\therefore 1*2\neq 2*1$; where $1,\ 2\in\mathbf{Z}$

Hence, the operation $*$ is not commutative.

Also,

$(1*2)*3=(1-2)*3=-1*3=-1-3=-4$

$1*(2*3)=1*(2-3)=1*-1=1-(-1)=2$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in\mathbf{Z}$

Hence, the operation $*$ is not associative.

ii. On $\mathbf{Q}$, define $a*b=ab+1$

$ab=ba$ for all $a,b\in Q$

$\Rightarrow ab+1=ba+1$ for all $a,b\in Q$

$\Rightarrow a*b=b*a$ for all $a,b\in Q$

Hence, the operation $*$ is commutative.

$(1*2)*3=(1\times 2+1)*3=3*3=3\times 3+1=10$

$1*(2*3)=1*(2\times 3+1)=1*7=1\times 7+1=8$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in\mathbf{Q}$

Hence, the operation $*$ is not associative.

iii. On $\mathbf{Q}$, define $a*b=\frac{ab}{2}$

$ab=ba$ for all $a,b\in Q$

$\Rightarrow \frac{ab}{2}=\frac{ab}{2}$ for all $a,b\in Q$

$\Rightarrow a*b=b*a$ for all $a,b\in Q$

Hence, the operation $*$ is commutative.

$$(a*b)*c=\left(\frac{ab}{2}\right)*c=\frac{\left(\frac{ab}{2}\right)c}{2}=\frac{abc}{4}$$

And

$$a*(b*c)=a*\left(\frac{bc}{2}\right)=\frac{a\left(\frac{bc}{2}\right)}{2}=\frac{abc}{4}$$

$\therefore (a*b)*c=a*(b*c)$ where $a,b,c\in\mathbf{Q}$

Hence, the operation $*$ is associative.

iv. On $\mathbf{Z}^+$, define $a*b=2^{ab}$

$ab=ba$ for all $a,b\in Z$

$\Rightarrow 2^{ab}=2^{ba}$ for all $a,b\in Z$

$\Rightarrow a*b=b*a$ for all $a,b\in Z$

Hence, the operation $*$ is commutative.

$(1*2)*3=2^{1\times 2}*3=4*3=2^{4\times 3}=2^{12}$

$1*(2*3)=1*2^{2\times 3}=1*2^6=1*64=2^{64}$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in\mathbf{Z}^+$

Hence, the operation $*$ is not associative.

v. On $\mathbf{Z}^+$, define $a*b=a^b$

$1*2=1^2=1$

$2*1=2^1=2$

$\therefore 1*2\neq 2*1$ where $1,2,\in\mathbf{Z}^+$

Hence, the operation $*$ is not commutative.

$(2*3)*4=2^3*4=8*4=8^4=2^{12}$

$2*(3*4)=2*3^4=2*81=2^{81}$

$\therefore (2*3)*4\neq 2*(3*4)$ where $2,3,4\in\mathbf{Z}^+$

Hence, the operation $*$ is not associative.

vi. On $\mathbf{R}-\{-1\}$, define $a*b=\frac{a}{b+1}$

$$1*2=\frac{1}{2+1}=\frac{1}{3}$$

$$2*1=\frac{2}{1+1}=\frac{2}{2}=1$$

$\therefore 1*2 \neq 2*1$ where $1,2,\in \mathbf{R}-\{-1\}$

Hence, the operation $*$ is not commutative.

$$(1*2)*3=\frac{1}{3}*3=\frac{\frac{1}{3}}{3+1}=\frac{1}{12}$$

$$1*(2*3)=1*\frac{2}{3+1}=1*\frac{2}{4}=1*\frac{1}{2}=\frac{1}{\frac{1}{2}+1}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$$

$\therefore (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in \mathbf{R}-\{-1\}$

Hence, the operation $*$ is not associative.

Question 3:

Consider the binary operation $\wedge$ on the set $\{1,2,3,4,5\}$ defined by $a \wedge b=\min\{a,b\}$. Write the operation table of the operation $\wedge$.

Solution:

The binary operation $\wedge$ on the set $\{1,2,3,4,5\}$ is defined by $a \wedge b=\min\{a,b\}$ for all $a,b \in \{1,2,3,4,5\}$.

The operation table for the given operation $\wedge$ can be given as:

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4:

Consider a binary operation $*$ on the set $\{1,2,3,4,5\}$ given by the following multiplication table.

i. Compute $(2*3)*4$ and $2*(3*4)$

ii. Is $*$ commutative?

iii. Compute $(2*3)*(4*5)$.

(Hint: Use the following table)

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1

3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution:

$(2*3)*4 = 1*4 = 1$

i. $2*(3*4) = 2*1 = 1$

ii. For every $a, b \in \{1,2,3,4,5\}$, we have $a*b = b*a$. Therefore, $*$ is commutative.

iii. $(2*3)*(4*5)$

$(2*3) = 1 \text{ and } (4*5) = 1$

$\therefore (2*3)*(4*5) = 1*1 = 1$

Question 5:

Let $*'$ be the binary operation on the set $\{1,2,3,4,5\}$ defined by $a*'b =$ H.C.F. of a and b. Is the operation $*'$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.

Solution:

The binary operation on the set $\{1,2,3,4,5\}$ is defined by $a*'b =$ H.C.F. of a and b.
The operation table for the operation $*'$ can be given as:

$*'$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

The operation table for the operations $*'$ and $*$ are same.
operation $*'$ is same as operation $*$.

Question 6:

Let $*$ be the binary operation on N defined by $a*b =$ L.C.M. of a and b. Find

i. $5*7, 20*16$
ii. Is $*$ commutative?
iii. Is $*$ associative?
iv. Find the identity of $*$ in N
v. Which elements of N are invertible for the operation $*$?

Solution:

The binary operation on N is defined by $a*b=$ L.C.M. of a and b.

i. $5*7=$L.C.M of 5 and $7=35$
$20*16=$LCM of 20 and $16=80$

ii. L.C.M. of a and $b=$LCM of b and a for all $a,b\in N$
$\therefore a*b=b*a$
Operation $*$ is commutative.

iii. For $a,b,c\in N$
$(a*b)*c=$ (L.C.M. of a and b)$*c=$ L.C.M. of a,b,c
$a*(b*c)=a*$(L.C.M. of b and c)= L.C.M. of a,b,c
$\therefore (a*b)*c=a*(b*c)$
Operation $*$ is associative.

iv. L.C.M. of a and $1=a=$ L.C.M. of 1 and a for all $a\in N$
$a*1=a=1*a$ for all $a\in N$
Therefore, 1 is the identity of $*$ in N.

v. An element a in N is invertible with respect to the operation $*$ if there exists an element b in N, such that $a*b=e=b*a$
$e=1$
L.C.M. of a and $b=1=$LCM of b and a possible only when a and b are equal to 1.
1 is the only invertible element of N with respect to the operation $*$.

Question 7:

Is $*$ defined on the set $\{1,2,3,4,5\}$ by $a*b=$ LCM of a and b a binary operation? Justify your answer.

Solution:

The operation $*$ on the set $\{1,2,3,4,5\}$ is defined by $a*b=$ LCM of a and b.
The operation table for the operation $*'$ can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

$3*2=2*3=6\notin A,$
$5*2=2*5=10\notin A,$
$3*4=4*3=12\notin A,$
$3*5=5*3=15\notin A,$
$4*5=5*4=20\notin A$
The given operation $*$ is not a binary operation.

Question 8:
Let $*$ be the binary operation on N defined by $a*b$= H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N?

Solution:
The binary operation $*$ on N defined by $a*b$= H.C.F. of a and b.
$\therefore a*b=b*a$
Operation $*$ is commutative.

For all $a,b,c\in N$,
$(a*b)*c$ = (HCF of a and b)$*c$ = HCF of a,b,c
$a*(b*c)$ = $a*$(HCF. of b and c)= HCF of a,b,c
$\therefore (a*b)*c=a*(b*c)$
Operation $*$ is associative.

$e\in N$ will be the identity for the operation $*$ if $a*e=a=e*a$ for all $a\in N$. But this relation is not true for any $a\in N$.
Operation $*$ does not have any identity in N.

Question 9:
Let $*$ be the binary operation on Q of rational numbers as follows:
i. $a*b=a-b$
ii. $a*b=a^2+b^2$
iii. $a*b=a+ab$
iv. $a*b=(a-b)^2$
v. $a+b=\frac{ab}{4}$
vi. $a*b=ab^2$

Find which of the binary operations are commutative and which are associative.

Solution:

i. On Q, the operation * is defined as $a*b=a-b$

$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}-\frac{1}{3}=\frac{3-2}{3}=\frac{1}{6}$

And

$\frac{1}{3}*\frac{1}{2}=\frac{1}{3}-\frac{1}{2}=\frac{2-3}{6}=\frac{-1}{6}$

$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)\neq\left(\frac{1}{3}*\frac{1}{2}\right)$ where $\frac{1}{2},\frac{1}{3}\in Q$

Operation * is not commutative.

$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}-\frac{1}{3}\right)*\frac{1}{4}=\frac{1}{6}*\frac{1}{4}=\frac{1}{6}-\frac{1}{4}=\frac{2-3}{12}=\frac{-1}{12}$

$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{2}*\frac{1}{12}=\frac{1}{2}-\frac{1}{12}=\frac{6-1}{12}=\frac{5}{12}$

$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)$ where $\frac{1}{2},\frac{1}{3},\frac{1}{4}\in Q$

Operation * is not associative.

ii. On Q, the operation * is defined as $a*b=a^2+b^2$

For $a,b\in Q$

$a*b=a^2+b^2=b^2+a^2=b*a$

$\therefore a*b=b*a$

Operation * is commutative.

$(1*2)*3=(1^2+2^2)*3=(1+4)*3=5*3=5^2+3^2=25+9=34$

$1*(2*3)=1*(2^2+3^2)=1*(4+9)=1*13=1^2+13^2=1+169=170$

$\therefore(1*2)*3\neq1*(2*3)$ where $1,2,3\in Q$

Operation * is not associative.

iii. On Q, the operation * is defined as $a*b=a+ab$

$1*2=1+1\times2=1+2=3$

$2*1=2+2\times1=2+2=4$

$\therefore 1*2\neq2*1$ where $1,2\in Q$

Operation * is not commutative.

$(1*2)*3=(1+1\times2)*3=3*3=3+3\times3=3+9=12$

$1*(2*3)=1*(2+2\times3)=1*8=1+1\times8=1+8=9$

$\therefore(1*2)*3\neq1*(2*3)$ where $1,2,3\in Q$

Operation * is not associative.

iv. On Q, the operation $*$ is defined as $a*b=(a-b)^2$

For $a,b\in Q$

$a*b=(a-b)^2$

$b*a=(b-a)^2=\left[-(a-b)\right]^2=(a-b)^2$

$\therefore a*b=b*a$

Operation $*$ is commutative.

$(1*2)*3=(1-2)^2*3=(-1)^2*3=1*3=(1-3)^2=(-2)^2=4$

$1*(2*3)=1*(2-3)^2=1*(-1)^2=1*1=(1-1)^2=0$

$\therefore (1*2)*3\neq 1*(2*3)$ where $1,2,3\in Q$

Operation $*$ is not associative.

v. On Q, the operation $*$ is defined as $a+b=\dfrac{ab}{4}$

For $a,b\in Q$

$a*b=\dfrac{ab}{4}=\dfrac{ba}{4}=b*a$

$\therefore a*b=b*a$

Operation $*$ is commutative.

For $a,b,c\in Q$

$$(a*b)*c=\frac{ab}{4}*c=\frac{\frac{ab}{4}\cdot c}{4}=\frac{abc}{16}$$

$$a*(b*c)=a*\frac{ab}{4}=\frac{a\cdot\frac{ab}{4}}{4}=\frac{abc}{16}$$

$\therefore (a*b)*c=a*(b*c)$ where $a,b,c\in Q$

Operation $*$ is associative.

vi. On Q, the operation $*$ is defined as $a*b=ab^2$

$$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2=\frac{1}{2}\cdot\frac{1}{9}=\frac{1}{18}$$

$$\frac{1}{3}*\frac{1}{2}=\frac{1}{3}\cdot\left(\frac{1}{2}\right)^2=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$$

$\therefore \left(\dfrac{1}{2}*\dfrac{1}{3}\right)\neq\left(\dfrac{1}{3}*\dfrac{1}{2}\right)$ where $\dfrac{1}{2},\dfrac{1}{3}\in Q$

Operation $*$ is not commutative.

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2\right)*\frac{1}{4}=\frac{1}{18}*\frac{1}{4}=\frac{1}{18}\cdot\left(\frac{1}{4}\right)^2=\frac{1}{18\times16}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}\cdot\left(\frac{1}{4}\right)^2\right)=\frac{1}{2}*\frac{1}{48}=\frac{1}{2}\cdot\left(\frac{1}{48}\right)^2=\frac{1}{2\times(48)^2}$$

$$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right) \quad \text{where } \frac{1}{2},\frac{1}{3},\frac{1}{4}\in Q$$

Operation $*$ is not associative.

Operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Question 10:

Find which of the operations given above has identity.

Solution:

An element $e\in Q$ will be the identity element for the operation $*$ if

$a*e=a=e*a,$ for all $a\in Q$

$$a*b=\frac{ab}{4}$$
$$\Rightarrow a*e=a$$
$$\Rightarrow \frac{ae}{4}=a$$
$$\Rightarrow e=4$$

Similarly, it can be checked for $e*a=a$, we get $e=4$ is the identity.

Question 11:

$A=N\times N$ and $*$ be the binary operation on A defined by $(a,b)*(c,d)=(a+c,b+d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A, if any.

Solution:

$A=N\times N$ and $*$ be the binary operation on A defined by

$$(a,b)*(c,d)=(a+c,b+d)$$
$$(a,b)*(c,d)\in A$$
$$a,b,c,d\in N$$
$$(a,b)*(c,d)=(a+c,b+d)$$
$$(c,d)*(a,b)=(c+a,d+b)=(a+c,b+d)$$
$$\therefore (a,b)*(c,d)=(c,d)*(a,b)$$

Operation $*$ is commutative.

Now, let $(a,b),(c,d),(e,f) \in A$

$a,b,c,d,e,f \in N$

$[(a,b)*(c,d)]*(e,f) = (a+c,b+d)*(e,f) = (a+c+e,b+d+f)$

$(a,b)*[(c,d)*(e,f)] = (a,b)*(c+e,d+f) = (a+c+e,b+d+f)$

$\therefore [(a,b)*(c,d)]*(e,f) = (a,b)*[(c,d)*(e,f)]$

Operation $*$ is associative.

An element $e = (e_1, e_2) \in A$ will be an identity element for the operation $*$ if $a+e = a = e*a$ for all $a = (a_1, a_2) \in A$ i.e., $(a_1+e_1, a_2+e_2) = (a_1, a_2) = (e_1+a_1, e_2+a_2)$, which is not true for any element in A.

Therefore, the operation $*$ does not have any identity element.

Question 12:

State whether the following statements are true or false. Justify.

i. For an arbitrary binary operation $*$ on a set N, $a*a = a$ for all $a \in N$.

ii. If $*$ is a commutative binary operation on N, then $a*(b*c) = (c*b)*a$

Solution:

i. Define operation $*$ on a set N as $a*a = a$ for all $a \in N$.
In particular, for $a = 3$,
$3*3 = 9 \neq 3$
Therefore, statement (i) is false.

ii. R.H.S. $= (c*b)*a$
$= (b*c)*a$ [$*$ is commutative]
$= a*(b*c)$ [Again, as $*$ is commutative]
= L.H.S.
$\therefore a*(b*c) = (c*b)*a$
Therefore, statement (ii) is true.

Question 13:

Consider a binary operation $*$ on N defined as $a*b = a^3 + b^3$. Choose the correct answer.

A. Is * both associative and commutative?
B. Is * commutative but not associative?
C. Is * associative but not commutative?
D. Is * neither commutative nor associative?

Solution:

On N, operation $*$ is defined as $a*b = a^3 + b^3$.

For all $a, b \in N$

$a*b = a^3 + b^3 = b^3 + a^3 = b*a$

Operation $*$ is commutative.

$(1*2)*3 = (1^3 + 2^3)*3 = (1+8)*3 = 9*3 = 9^3 + 3^3 = 729 + 27 = 756$

$1*(2*3) = 1*(2^3 + 3^3) = 1*(8+27) = 1*35 = 1^3 + 35^3 = 1 + 42875 = 42876$

$\therefore (1*2)*3 \neq 1*(2*3)$ Operation $*$ is not associative.

Therefore, Operation $*$ is commutative, but not associative.

The correct answer is B.