NCERT Solutions Class 12 Maths Chapter 1 Miscellaneous Exercise

Question 1:

Let $f: R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g: R \to R$ such that $gof = fog = I_R$.

Solution:

$f: R \to R$ is defined as $f(x) = 10x + 7$

For one-one:

$f(x) = f(y)$ where $x, y \in R$

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore f$ is one-one.

For onto:

$y \in R$, Let $y = 10x + 7$

$\Rightarrow x = \dfrac{y-7}{10} \in R$

For any $y \in R$, there exists $x = \dfrac{y-7}{10} \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore f$ is onto.

Thus, f is an invertible function.

Let us define $g: R \to R$ as $g(y) = \dfrac{y-7}{10}$.

Now,

$$gof(x) = g(f(x)) = g(10x + 7) = \frac{(10x + 7) - 7}{10} = \frac{10x}{10} = 10$$

And,

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore gof = I_R$ and $fog = I_R$

Hence, the required function $g: R \to R$ as $g(y) = \dfrac{y-7}{10}$.

Question 2:

Let $f : W \to W$ be defined as $f(n) = n - 1$, if is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Solution:

$f : W \to W$ is defined as $f(n) = \begin{Bmatrix} n-1, \text{ If } n \text{ is odd} \\ n+1, \text{ If } n \text{ is even} \end{Bmatrix}$

For one-one:

$f(n) = f(m)$

If n is odd and m is even, then we will have $n - 1 = m + 1$.

$\Rightarrow n - m = 2$

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

∴Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m)$

$\Rightarrow n - 1 = m - 1$

$\Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m)$

$\Rightarrow n + 1 = m + 1$

$\Rightarrow n = m$

$\therefore f$ is one-one.

For onto:

Any odd number $2r + 1$ in co-domain N is the image of $2r$ in domain N and any even number $2r$ in co-domain N is the image of $2r + 1$ in domain N.

$\therefore f$ is onto.

f is an invertible function.

Let us define $g : W \to W$ as $f(m) = \begin{Bmatrix} m-1, \text{ If } m \text{ is odd} \\ m+1, \text{ If } m \text{ is even} \end{Bmatrix}$

When r is odd

$gof(n) = g(f(n)) = g(n-1) = n - 1 + 1 = n$

When r is even

$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$

When m is odd

$fog(n) = f(g(m)) = f(m-1) = m-1+1 = m$

When m is even

$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$

$\therefore gof = I_W$ and $fog = I_W$

f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f. inverse of f is f itself.

Question 3:

If $f : R \to R$ be defined as $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Solution:

$f : R \to R$ is defined as $f(x) = x^2 - 3x + 2$.

$f(f(x)) = f(x^2 - 3x + 2)$

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) + (-3x^2 + 9x - 6) + 2$

$= x^4 - 6x^3 + 10x^2 - 3x$

Question 4:

Show that function $f : R \to \{x \in R : -1 < x < 1\}$ be defined by $f(x) = \frac{x}{1+|x|}, x \in R$ is one-one and onto function.

Solution:

$f : R \to \{x \in R : -1 < x < 1\}$ is defined by $f(x) = \frac{x}{1+|x|}, x \in R$.

For one-one:

$f(x) = f(y)$ where $x, y \in R$

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

If x is positive and y is negative,

$$\frac{x}{1+|x|} = \frac{y}{1+|y|}$$
$$\Rightarrow 2xy = x - y$$

Since, x is positive and y is negative,

$x > y \Rightarrow x - y > 0$

$2xy$ is negative.

$2xy \neq x - y$

Case of x being positive and y being negative, can be ruled out.

$\therefore$ x and y have to be either positive or negative.

If x and y are positive,

$$f(x) = f(y)$$
$$\Rightarrow \frac{x}{1+x} = \frac{y}{1+y}$$
$$\Rightarrow x - xy = y - xy$$
$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For onto:

Let $y \in R$ such that $-1 < y < 1$.

If x is negative, then there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If x is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|}=\frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)}=\frac{y}{1-y+y}=y$$

$\therefore f$ is onto.

Hence, f is one-one and onto.

Question 5:

Show that function $f:R\rightarrow R$ be defined by $f(x)=x^3$ is injective.

Solution:

$f:R\rightarrow R$ is defined by $f(x)=x^3$

For one-one:

$f(x)=f(y)$ where $x,y\in R$

$x^3=y^3$.......................(1)

We need to show that $x=y$

Suppose $x\neq y$, their cubes will also not be equal.

$\Rightarrow x^3\neq y^3$

This will be a contradiction to (1).

$\therefore x=y$. Hence, f is injective.

Question 6:

Give examples of two functions $f:N\rightarrow Z$ and $g:Z\rightarrow Z$ such that gof is injective but g is not injective.

(Hint: Consider $f(x)=x$ and $g(x)=|x|$)

Solution:

Define $f:N\rightarrow Z$ as $f(x)=x$ and $g:Z\rightarrow Z$ as $g(x)=|x|$

Let us first show that g is not injective.

$(-1)=|-1|=1$

$(1)=|1|=1$

$\therefore (-1)=g(1),\quad$ but $-1\neq 1$

$\therefore g$ is not injective.

$gof : N \to Z$ is defined as $gof(x) = g(f(x)) = g(x) = |x|$

$x, y \in N$ such that $gof(x) = gof(y)$

$\Rightarrow |x| = |y|$

Since $x, y \in N$, both are positive.

$\therefore |x| = |y|$

$\Rightarrow x = y$

$\therefore gof$ is injective.

Question 7:

Given examples of two functions $f : N \to N$ and $g : N \to N$ such that gof is onto but f is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x - 1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$)

Solution:

Define $f : N \to Z$ as $f(x) = x + 1$ and $g : Z \to Z$ as $g(x) = \begin{cases} x - 1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$

Let us first show that g is not onto.

Consider element 1 in co-domain N. This element is not an image of any of the elements in domain N.

$\therefore f$ is not onto.

$g : N \to N$ is defined by

$gof(x) = g(f(x)) = g(x+1) = x + 1 - 1 = x$ $\quad [x \in N \Rightarrow x + 1 > 1]$

For $y \in N$, there exists $x = y \in N$ such that $gof(x) = y$.

$\therefore gof$ is onto.

Question 8:

Given a non-empty set X, consider $P(X)$ which is the set of all subsets of X.

Define the relation R in $P(X)$ as follows:

For subsets A, B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify you answer.

Solution:

Since every set is a subset of itself, ARA for all $A \in P(X)$.
$\therefore R$ is reflexive.

Let $ARB \Rightarrow A \subset B$
This cannot be implied to $B \subset A$.
If $A = \{1,2\}$ and $B = \{1, 2,3\}$, then it cannot be implied that B is related to A.
$\therefore R$ is not symmetric.

If ARB and BRC, then $A \subset B$ and $B \subset C$.
$\Rightarrow A \subset C$
$\Rightarrow ARC$
$\therefore R$ is transitive.

R is not an equivalence relation as it is not symmetric.

Question 9:

Given a non-empty set X, consider the binary operation $*: P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \ \forall A, B$ in $P(X)$ is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation $*$.

Solution:

$P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \ \forall A, B$ in $P(X)$

$A \cap X = A = X \cap A$ for all $A \in P(X)$

$\Rightarrow A * X = A = X * A$ for all $A \in P(X)$
X is the identity element for the given binary operation $*$.

An element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that
$A * B = X = B * A$ [As X is the identity element]

Or

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

X is the only invertible element in $P(X)$ with respect to the given operation $*$.

Question 10:

Find the number of all onto functions from the set $\{1,2,3,\ldots,n\}$ to itself.

Solution:

Onto functions from the set $\{1,2,3,\ldots,n\}$ to itself is simply a permutation on n symbols $1,2,3,\ldots,n$.

Thus, the total number of onto maps from $\{1,2,3,\ldots,n\}$ to itself is the same as the total number of permutations on n symbols $1,2,3,\ldots,n$, which is $n!$.

Question 11:

Let $S=\{a,b,c\}$ and $T=\{1,\ 2,3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

i. $F=\{(a,3),(b,2),(c,1)\}$

ii. $F=\{(a,2),(b,1),(c,1)\}$

Solution: $S=\{a,b,c\},T=\{1,\ 2,3\}$

i. $F:S\to T$ is defined by $F=\{(a,3),(b,2),(c,1)\}$

$\Rightarrow F(a)=3,\ F(b)=2,\ F(c)=1$

Therefore, $F^{-1}:T\to S$ is given by $F^{-1}=\{(3,a),(2,b),(1,c)\}$

ii. $F:S\to T$ is defined by $F=\{(a,2),(b,1),(c,1)\}$

Since, $F(b)=F(c)=1$, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exists.

Question 12:

Consider the binary operations $*$: $R\times R\to R$ and $o:R\times R\to R$ defined as $a*b=|a-b|$ and $aob=a,\ \forall a,b\in R$. Show that $*$ is commutative but not associative o is associative but not commutative. Further, show that $\forall a,b,c\in R$, $a*(boc)=(a*b)o(a*c)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

Solution:

It is given that $*$: $R\times R\to R$ and $o:R\times R\to R$ defined as $a*b=|a-b|$ and $aob=a,\ \forall a,b\in R$.

For $a,b\in R$, we have $a*b=|a-b|$ and $b*a=|b-a|=|-(a-b)|=|a-b|$

$\therefore a*b=b*a$

$\therefore$ The operation $*$ is commutative.

$$(1*2)*3=(|1-2|)*3=1*3=|1-3|=2$$
$$1*(2*3)=1*(|2-3|)=1*1=|1-1|=0$$

$\therefore (1*2)*3 \neq 1*(2*3)$ where $1,2,3 \in R$

$\therefore$ The operation * is not associative.

Now, consider the operation o:

It can be observed that $1o2=1$ and $2o1=2$.

$\therefore 1o2 \neq 2o1$ (where $1,2 \in R$)

$\therefore$ The operation o is not commutative.

Let $a,b,c \in R$. Then, we have:

$$(aob)oc = aoc = a$$
$$ao(boc) = aob = a$$
$$\Rightarrow (aob)oc = ao(boc)$$

$\therefore$ The operation o is associative.

Now, let $a,b,c \in R$, then we have:

$$a*(boc) = a*b = |a-b|$$
$$(a*b)o(a*c) = (|a-b|)o(|a-c|) = |a-b|$$

Hence, $a*(boc) = (a*b)o(a*c)$

Now,

$$1o(2*3) = 1o(|2-3|) = 1o1 = 1$$
$$(1o2)*(1o3) = 1*1 = |1-1| = 0$$

$\therefore 1o(2*3) \neq (1o2)*(1o3)$ where $1,2,3 \in R$

$\therefore$ The operation o does not distribute over $*$.

Question 13:

Given a non - empty set X, let $*: \mathrm{P}(X)\times \mathrm{P}(X)\to \mathrm{P}(X)$ be defined as $A*B=(A-B)\cup(B-A)$, $\forall A,B\in P(X)$. Show that the empty set Φ is the identity for the operation $*$ and all the elements A of $P(X)$ are invertible with $A^{-1}=A$.

(Hint: $(A-\Phi)\cup(\Phi-A)=A$ and $(A-A)\cup(A-A)=A*A=\Phi$).

Solution:

It is given that $*: \mathrm{P}(X)\times \mathrm{P}(X)\to \mathrm{P}(X)$ is defined as $A*B=(A-B)\cup(B-A)$, $\forall A,B\in P(X)$

$A\in P(X)$ then,

$A*\Phi=(A-\Phi)\cup(\Phi-A)=A\cup\Phi=A$

$\Phi*A=(\Phi-A)\cup(A-\Phi)=\Phi\cup A=A$

$\therefore A*\Phi=A=\Phi*A$ for all $A\in P(X)$

Φ is the identity for the operation $*$.

Element $A\in P(X)$ will be invertible if there exists $B\in P(X)$ such that

$A*B=\Phi=B*A$ [As Φ is the identity element]

$A*A=(A-A)\cup(A-A)=\Phi\cup\Phi=\Phi$ for all $A\in P(X)$.

All the elements A of $P(X)$ are invertible with $A^{-1}=A$.

Question 14:

Define a binary operation $*$ on the set $\{0,1,2,3,4,5\}$ as

$$a+b=\begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6 & \text{if } a+b\geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a\neq 0$ of the set is invertible with $6-a$ being the inverse of a.

Solution:

Let $X=\{0,1,2,3,4,5\}$

The operation $*$ is defined as $a+b=\begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b\geq 6 \end{cases}$

An element $e\in X$ is the identity element for the operation $*$, if $a*e=a=e*a \quad \forall a\in X$

For $a\in X$,

$$a*0=a+0=a \qquad [a\in X\Rightarrow a+0<6]$$
$$0*a=0+a=a \qquad [a\in X\Rightarrow 0+a<6]$$
$$\therefore a*0=a=0*a \quad \forall a\in X$$

Thus, 0 is the identity element for the given operation $*$.

An element $a\in X$ is invertible if there exists $b\in X$ such that $a*b=0=b*a$.

i.e., $\begin{Bmatrix} a+b=0=b+a, & \text{if } a+b<6 \\ a+b-6=0=b+a-6 & \text{if } a+b\geq 6 \end{Bmatrix}$

$\Rightarrow a=-b$ or $b=6-a$

$X=\{0,1,2,3,4,5\}$ and $a,b\in X$. Then $a\neq -b$.

$\therefore b=6-a$ is the inverse of a for all $a\in X$.
Inverse of an element $a\in X$, $a\neq 0$ is $6-a$ i.e., $a-1=6-a$.

Question 15:

Let $A=\{-1,0,1,2\}$, $B=\{-4,-2,0,2\}$ and $f,g:A\rightarrow B$ be functions defined by x^2-x, $x\in A$ and $g(x)=2\left|x-\frac{1}{2}\right|-1,\ x\in A$. Are f and g equal?

Solution:

It is given that $A=\{-1,0,1,2\}$, $B=\{-4,-2,0,2\}$

Also, $f,g:A\rightarrow B$ is defined by x^2-x, $x\in A$ and $g(x)=2\left|x-\frac{1}{2}\right|-1,\ x\in A$.

$$f(-1)=(-1)^2-(-1)=1+1=2$$
$$g(-1)=2\left|(-1)-\frac{1}{2}\right|-1=2\left(\frac{3}{2}\right)-1=3-1=2$$
$$\Rightarrow f(-1)=g(-1)$$
$$f(0)=(0)^2-0=0$$
$$g(0)=2\left|0-\frac{1}{2}\right|-1=2\left(\frac{1}{2}\right)-1=1-1=0$$
$$\Rightarrow f(0)=g(0)$$

$f(1)=(1)^2-1=0$

$g(1)=2\left|1-\frac{1}{2}\right|-1=2\left(\frac{1}{2}\right)-1=1-1=0$

$\Rightarrow f(1)=g(1)$

$f(2)=(2)^2-2=2$

$g(2)=2\left|2-\frac{1}{2}\right|-1=2\left(\frac{3}{2}\right)-1=3-1=2$

$\Rightarrow f(2)=g(2)$

$\therefore f(a)=g(a) \quad \forall a\in A$

Hence, the functions f and g are equal.

Question 16:

Let $A=\{1,2,3\}$. Then number of relations containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is,

A. 1
B. 2
C. 3
D. 4

Solution:

The given set is $A=\{1,2,3\}$.

The smallest relation containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is given by,

$R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,1)\}$

This is because relation R is reflexive as $\{(1,1),(2,2),(3,3)\}\in R$.

Relation R is symmetric as $\{(1,2),(2,1)\}\in R$ and $\{(1,3)(3,1)\}\in R$.

Relation R is transitive as $\{(3,1),(1,2)\}\in R$ but $(3,2)\notin R$.

Now, if we add any two pairs $(3,2)$ and $(2,3)$ (or both) to relation R, then relation R will become transitive.
Hence, the total number of desired relations is one.
The correct answer is A.

Question 17:

Let $A=\{1,2,3\}$. Then number of equivalence relations containing $(1,2)$ is,

A. 1
B. 2
C. 3
D. 4

Solution:

The given set is $A=\{1,2,3\}$.

The smallest equivalence relation containing $(1,2)$ is given by;

$R_1=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

Now, we are left with only four pairs i.e., $(2,3),(3,2),(1,3)$ and $(3,1)$.

If we odd any one pair [say $(2,3)$] to R_1, then for symmetry we must add $(3,2)$. Also, for transitivity we are required to add $(1,3)$ and $(3,1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1,2)$ is two.
The correct answer is B.

Question 18:

Let $f:R\to R$ be the Signum Function defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$ and $g:R\to R$ be the greatest integer function given by $g(x)=[x]$, where $[x]$ is greatest integer less than or equal to x. Then does fog and gof coincide in $(0,1]$?

Solution:

It is given that $f:R\to R$ be the Signum Function defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$

Also $g:R\to R$ is defined as $g(x)=[x]$, where $[x]$ is greatest integer less than or equal to x.
Now let $x\in(0,1]$,

$[x]=1$ if $x=1$ and $[x]=0$ if $0<x<1$.

$$\therefore fog(x) = f(g(x)) = f([x]) = \begin{Bmatrix} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0,1) \end{Bmatrix} = \begin{Bmatrix} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0,1) \end{Bmatrix}$$

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(1) \qquad [x > 0] \\ &= [1] = 1 \end{aligned}$$

Thus, when $x \in (0,1)$, we have $fog(x) = 0$ and $gof(x) = 1$.

Hence, fog and gof does not coincide in $(0,1]$.

Question 19:

Number of binary operations on the set $\{a,b\}$ are

A. 10
B. 16
C. 20
D. 8

Solution:

A binary operation * on $\{a,b\}$ is a function from $\{a,b\} \times \{a,b\} \to \{a,b\}$

i.e., * is a function from $\{(a,a),(a,b),(b,a),(b,b)\} \to \{a,b\}$

Hence, the total number of binary operations on the set $\{a,b\}$ is $2^4 = 16$.

The correct answer is B.